Classification of operator algebraic conformal field theories in dimensions one and two

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Abstract

We formulate conformal field theory in the setting of algebraic quantum field theory as Haag-Kastler nets of local observable algebras with diffeomorphism covariance on the two-dimensional Minkowski space. We then obtain a decomposition of a two-dimensional theory into two chiral theories. We give the first classification result of such chiral theories with representation theoretic invariants. That is, we use the central charge as the first invariant, and if it is less than 1, we obtain a complete classification. Our classification list contains a new net which does not seem to arise from the known constructions such as the coset or orbifold constructions. We also present a classification of full two-dimensional conformal theories. These are joint works with Roberto Longo.

1 Introduction

Our main results, together with R. Longo, are classification results for conformal field theories, in the operator algebraic approach. We first briefly describe our basic framework for quantum field theory and its relation to a more conventional approach based on Wightman axioms using operator-valued distributions.

Our framework is called *algebraic quantum field theory* or *local quantum physics*, and its standard textbook is [14] by R. Haag. We first explain our axiomatic setting on the 4-dimensional Minkowski space, although we will later work on lower dimensional spacetime. Recently, several attempts have been made on studies on curved spacetime or even noncommutative spacetime, but we will not deal with such topics in this review.

In our setting, a physical system is described by a family of operator algebras $\mathcal{A}(O)$ on a fixed Hilbert space H, where O is a bounded region in the Minkowski space. As such a region O, we consider only double cones, which are of the form $(x+V_+)\cap(y+V_-)$, where $x, y \in \mathbb{R}^4$ and

$$V_\pm = \{z = (z_0, z_1, z_2, z_3) \in \mathbb{R}^4 \mid z_0^2 - z_1^2 - z_2^2 - z_3^2 > 0, \pm z_0 > 0\}.$$

We assume that we have a von Neumann algebra $\mathcal{A}(O)$ acting on H for each double cone O and the following properties hold. (An algebra of bounded linear operators on

a Hilbert space is called a von Neumann algebra if it is closed under the $*$-operation and weak-operator topology.)

1. (Isotony) For $O_1 \subset O_2$, we have $\mathcal{A}(O_1) \subset \mathcal{A}(O_2)$.
2. (Locality) If O_1 and O_2 are spacelike separated, then elements in $\mathcal{A}(O_1)$ and $\mathcal{A}(O_2)$ commute.
3. (Poincaré Covariance) There exists a unitary representation U of the universal covering of the restricted Poincaré group satisfying $\mathcal{A}(gO) = U_g \mathcal{A}(O) U_g^*$.
4. (Vacuum) We have a unit vector $\Omega \in H$, unique up to phase, satisfying $U_g \Omega = \Omega$ for all elements g in the restricted Poincaré group and $\bigcup_O \mathcal{A}(O)\Omega$ is dense in H.
5. (Spectrum Condition) If we restrict the representation U to the translation subgroup, its spectrum is contained in the closure of V_+.

The isotony axioms simply states that we have more observables for a larger region. The locality axiom means that if we have two spacelike separated regions, then we have no interactions between them even at a speed of light, so the two operators taken from the two regions mutually commute. It is also called the Einstein causality. Covariance means that a "spacetime symmetry" acts as a symmetry of the family of operator algebras. We will later use a higher spacetime symmetry than restricted Poincaré group. The vector Ω is called a vacuum vector and it gives a vacuum state. The spectrum condition means stability.

If we denote the set of the elements that are spacelike separated with all the elements of a region D by $D^\perp$, then we have $O^{\perp\perp} = O$ for a double cone O. This is why we use only double cones. For a general region D, we could define $\mathcal{A}(D)$ as the von Neumann algebra generated by $\mathcal{A}(O)$ for all double cones O contained in D.

Since the set of double cones is directed with respect to inclusions, we often say that the family $\mathcal{A}(O)$ is a *net of von Neumann algebras*. We also say a *net of factors*, if each von Neumann algebra $\mathcal{A}(O)$ has a trivial center, which is often the case in the lower dimensional spacetime as below, since such a von Neumann algebra is called a factor. In many such cases, the local algebras are all isomorphic, so each local algebra itself does not contain physical information about the system. A basic idea is that all information about a certain physical system is contained in such a net $\mathcal{A}(O)$.

From a mathematical viewpoint, such a net of von Neumann algebras is simply a family of operator algebras subject to certain set of axioms, so we can study classification theory of such families of operator algebras up to an obvious notion of isomorphism. A useful and important tool for such a study is a representation theory of a net of von Neumann algebras.

A basic tool to study a net of von Neumann algebras is its representation theory formulated by Doplicher-Haag-Roberts (DHR) [7]. Each operator algebra $\mathcal{A}(O)$ acts on a fixed Hilbert space from the beginning, but we can also consider a representation of a family of operator algebras on a different Hilbert space where we do not have a vacuum vector any more. A basic idea of Doplicher-Haag-Roberts is that if we assume a nice

condition called the Haag duality and select a nice class of representations with their criterion, then each such representation is realized, up to unitary equivalence, as a certain endomorphism of (the norm closure of) $\bigcup_O \mathcal{A}(O)$. Such an endomorphism is often called a DHR endomorphism. An important feature of endomorphisms is that they can be composed. This composition gives an operation in the set of DHR endomorphisms which plays a role of a tensor product. Through this operation (and others), mathematical structure of DHR endomorphisms becomes quite similar to that of unitary representations of a compact group, and it gives a C^*-tensor category.

We briefly mention a relation of the above approach to a more conventional one based on the Wightman axioms. In the setting of the Wightman axioms, one considers a family of operator-valued distributions $\{\phi_j(x)\}$ on the Minkowski space. If we have such a family at the beginning, then, roughly speaking, we apply smooth functions supported in O to these distributions, apply bounded functional calculus to the resulting (unbounded) operators, and let $\mathcal{A}(O)$ be the von Neumann algebra generated by these bounded operators. In this way, we should obtain a local net of von Neumann algebras. If we start with a local net $\mathcal{A}(O)$ of von Neumann algebras, we should obtain operator valued distributions $\{\phi_j(x)\}$ through a certain limiting procedure in which bounded regions O shrink to one point x. It is believed that the approach based on the Wightman axioms and the one based on local nets of von Neumann algebras are essentially equivalent, and there have been many works which study under what conditions we obtain one from the other, but the exact relations between the two approaches have not been fully understood yet.

2 Full and chiral algebraic conformal quantum field theories

The above general framework in the previous section obviously works on a Minkowski space of any dimension. We now specialize on the 2-dimensional Minkowski space $\mathcal{M}$ and require higher symmetry than the general Poincaré covariance. This is our approach to conformal field theory. Then through a chiral decomposition of a full algebraic conformal quantum field theory, we obtain a chiral algebraic conformal quantum field theory which is now described as a one-dimensional net of factors. After such a general description, we will briefly mention a relation to vertex (operator) algebras, which give another mathematical approach to chiral conformal field theories.

We now work on a two-dimensional Minkowski space $\mathcal{M}$ where we use t and x for the time and space coordinates, respectively. We have a von Neumann algebra $\mathcal{A}(O)$ on a fixed Hilbert space H for each double cone O in this Minkowski space M as above. We set $L_\pm = \{t \pm x = 0\}$ and each double cone is a direct product $I_+ \times I_-$, where $I_\pm$ are bounded intervals in $L_\pm$, respectively. We consider the Möbius group $PSL(2, \mathbb{R})$ which acts on $\mathbb{R} \cup \{\infty\}$ as linear fractional transformations. In this way, we obtain a local action of the universal covering group $\overline{PSL}(2, \mathbb{R})$ on $\mathbb{R}$. We impose the following axioms for our net of von Neumann algebras $\mathcal{A}(O)$ on H and call such a net a *Möbius covariant net* of von Neumann algebras. (See [18] for more details.)

1. (Isotony) For $O_1 \subset O_2$, we have $\mathcal{A}(O_1) \subset \mathcal{A}(O_2)$.

2. (Locality) If O_1 and O_2 are spacelike separated, then elements in $\mathcal{A}(O_1)$ and $\mathcal{A}(O_2)$ commute.

3. (Möbius Covariance) There exists a unitary representation U of $\overline{PSL}(2,\mathbb{R}) \times \overline{PSL}(2,\mathbb{R})$ on H such that for every double cone O, we have $\mathcal{A}(gO) = U_g\mathcal{A}(O)U_g^*$ when $g \in W$ where W is a connected neighbourhood of the identity in $\overline{PSL}(2,\mathbb{R})\times \overline{PSL}(2,\mathbb{R})$ satisfying $gO \subset M$ for all $g \in W$.

4. (Vacuum) We have a unit vector $\Omega \in H$, unique up to phase, satisfying $U_g\Omega = \Omega$ for all elements g and $\bigcup_O \mathcal{A}(O)\Omega$ is dense in H.

5. (Positive energy) The one-parameter unitary subgroup of U corresponding to the time translation has a positive generator.

We now further strengthen the axiom of Möbius covariance as follows. Let G be the quotient of $\overline{PSL}(2,\mathbb{R}) \times \overline{PSL}(2,\mathbb{R})$ modulo the relation $(r_{2\pi}, r_{-2\pi}) = (\mathrm{id}, \mathrm{id})$. Then it turns out that our representation U as above gives a representation of this group G, due to the spacelike locality. We then find that our net $\mathcal{A}(O)$ extends to a local G-covariant net on the Einstein cylinder $\mathcal{E} = \mathbb{R} \times S^1$, which is the cover of the 2-torus obtained by lifting the time coordinate from S^1 to $\mathbb{R}$. We also have several consequences from the above set of axioms. See [18, Proposition 2.2], for example.

Let $\mathrm{Diff}(\mathbb{R})$ be the group of the orientation preserving diffeomorphisms which are smooth at infinity. Then this group naturally acts on $\mathcal{E}$ as a diffeomorphic action. Let $\mathrm{Conf}(\mathcal{E})$ be the group of global, orientation preserving conformal diffeomorphisms of $\mathcal{E}$. This group is generated by $\mathrm{Diff}(\mathbb{R}) \times \mathrm{Diff}(\mathbb{R})$ and G. If a Möbius covariant net $\mathcal{A}$ further satisifies the following axiom, we say that the net $\mathcal{A}$ is a *local conformal net*. This is the class we study.

(Diffeomorphism covariance) The unitary representation U of G extends to a projective unitary representation of $\mathrm{Conf}(\mathcal{E})$ such that the extended net on $\mathcal{E}$ is covariant. Furthermore, we have $U_gXU_g^* = X$ for $g \in \mathrm{Diff}(\mathbb{R}) \times \mathrm{Diff}(\mathbb{R})$, if $X \in \mathcal{A}(O)$ and g acts on O as identity.

This gives our framework for conformal quantum field theory. We study a net $\mathcal{A}(O)$ as a family of von Neumann algebras satisfying the above set of axioms. Such a family here is also called a *full algebraic conformal quantum field theory*. The DHR theory works in this setting perfectly.

Suppose we have a local conformal net $\mathcal{A}$ as above. Then for each bounded interval $I \subset L_+$, we set $\mathcal{A}_+(I) = \bigcap_J \mathcal{A}(I \times J)$. In this way, we have a family of von Neumann algebras $\mathcal{A}_+$ parameterized by bounded intervals I. We regard these von Neumann algebras as subalgebras of $B(H_+)$, where H_+ is the closure of $\bigcup_I \mathcal{A}_+(I)\Omega$. This family extends to a family $\mathcal{A}_+(I)$, where I is any open, nondense, nonempty, and connected set of $S^1 = \mathbb{R} \cup \{\infty\}$. (Such I is simply called an interval in S^1.) This family $\mathcal{A}_+(I)$ satisfies the following conditions. We may take these as axioms for such a family. (See [13] for more details.)

1. (Isotony) For $I_1 \subset I_2$, we have $\mathcal{A}_+(I_1) \subset \mathcal{A}_+(I_2)$.

2. (Locality) If I_1 and I_2 are disjoint, then elements in $\mathcal{A}_+(I_1)$ and $\mathcal{A}_+(I_2)$ commute.

3. (Diffeomorphism Covariance) There exists a projective unitary representation U of $\mathrm{Diff}(S^1)$ on H_+ such that for every interval I, we have $\mathcal{A}_+(gI) = U_g \mathcal{A}_+(I) U_g^*$. Furthermore, we have $U_g X U_g^* = X$ for $g \in \mathrm{Diff}(S^1)$, if $X \in \mathcal{A}(I)$ and g acts on I as identity.

4. (Vacuum) We have a unit vector $\Omega \in H_+$, unique up to phase, satisfying $U_g\Omega = \Omega$ for all elements g in the Möbius group $PSL(2, \mathbb{R})$ and $\bigcup_I \mathcal{A}_+(I)\Omega$ is dense in H_+.

5. (Positive energy) The one-parameter unitary subgroup of U corresponding to the rotation on S^1 has a positive generator.

A family $\mathcal{A}_+$ satisfying the above set of axioms is called a *chiral algebraic conformal quantum field theory*. We can similarly define $\mathcal{A}_-$. We then have an embedding $\mathcal{A}_+(I) \otimes \mathcal{A}_-(J) \subset \mathcal{A}(I \times J)$. The DHR theory works fine for a chiral algebraic conformal quantum field theory. Now, for two DHR endomorphisms ρ, σ, we have a unitary equivalence for $\rho\sigma$ and $\sigma\rho$, but we have a canonical unitary $\varepsilon(\rho, \sigma)$ implementing this unitary equivalence and this family ε of unitaries contains non-trivial information. This family ε is called a *braiding*. See [9], for example, for details.

Our classification method for full algebraic conformal quantum field theories $\mathcal{A}(O)$ consists of two steps. In the first step, we classify the chiral algebraic conformal quantum field theories $\mathcal{A}_\pm$. In the second step, we classify the embedding $\mathcal{A}_+(I) \otimes \mathcal{A}_-(J) \subset \mathcal{A}(I \times J)$, which is a non-trivial subfactor, usually.

3 Complete rationality and classification

Our basic idea for classification is that if we have a certain nice condition, generally called "amenability", a simple set of invariants related to representation theory should give a complete classification. We have given a general idea along this line in [16], so here we only briefly explain the condition called *complete rationality*, which was introduced in [19] and plays a role of amenability in classification theory.

Here we state complete rationality for a chiral algebraic conformal quantum field theory $\mathcal{A}(I)$, $I \subset S^1$. We also have a version for a full algebraic conformal quantum field theory and we refer the reader to [18] for the definition in such a setting. Consider a chiral algebraic conformal quantum field theory $\mathcal{A}$. Split S^1 into $2n$ intervals, and label them $I_1, I_2, \dots, I_{2n}$ in the counterclockwise order. Let μ_n be the Jones index of the subfactor $\mathcal{A}(I_1) \vee \mathcal{A}(I_3) \vee \cdots \vee \mathcal{A}(I_{2n-1}) \subset (\mathcal{A}(I_2) \vee \mathcal{A}(I_4) \vee \cdots \vee \mathcal{A}(I_{2n}))'$. (Note that we have this inclusion because of locality.) This number is independent of the way to split the circle. We remark that we automatically have $\mu_1 = 1$, which is called the *Haag duality*. Complete rationality consists of the following three conditions.

1. (Strong additivity) Remove one point from an interval I and label the resulting two intervals as I_1, I_2. Then we have $\mathcal{A}(I) = \mathcal{A}(I_1) \vee \mathcal{A}(I_2)$.

2. (Split property) Consider two intervals I_1, I_2 with $\bar{I}_1 \cap \bar{I}_2 = \emptyset$. Then the von Neumann algebra $\mathcal{A}(I_1) \vee \mathcal{A}(I_2)$ is naturally isomorphic to $\mathcal{A}(I_1) \otimes \mathcal{A}(I_2)$.

3. (Finiteness of the μ-index) We have $\mu_2 < \infty$.

The main results in [19] give the following two conditions under complete rationality.

1. We have only finitely many equivalence classes of irreducible DHR endomorphisms of the net $\mathcal{A}$.

2. The braiding naturally gives a unitary representations of $SL(2, \mathbb{Z})$ whose dimension is the number of the equivalence classes in (1).

This shows that the category of the DHR endomorphisms of the net $\mathcal{A}$ gives a modular tensor category, which plays an important role in theory of quantum invariants of 3-manifolds as in [31].

We next explain the first numerical invariant of a local conformal net, a central charge, of $\mathcal{A}$. Let $\mathcal{A}$ be a local conformal net. (Here we do not need complete rationality.) Then we have a projective unitary representation of $\mathrm{Diff}(S^1)$. Recall that the Virasoro algebra is the infinite dimensional Lie algebra generated by elements $\{L_n \mid n \in \mathbb{Z}\}$ and c with relations

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m,-n},$$

and $[L_n, c] = 0$. This is unique, non-trivial one-dimensional central extension of the Lie algebra of $\mathrm{Diff}(S^1)$. We now obtain a representation of the Virasoro algebra and then the central element c is mapped to a scalar. This value is the *central charge* of the net $\mathcal{A}$ and is also denoted by c. It has been shown by Friedan-Qiu-Shenker [11] that this central charge value is in

$$\{1 - 6/m(m+1) \mid m = 2, 3, 4, \dots\} \cup [1, \infty)$$

and the values $\{1 - 6/m(m+1) \mid m = 2, 3, 4, \dots\}$ have been realized by Goddard-Kent-Olive [12]. (The values in $[1, \infty)$ are easier to realize.) Jones has proved in his theory of index for subfactors [15] that the index value is in the set $\{4\cos^2 \pi/m \mid m = 3, 4, 5 \dots\} \cup [4, \infty]$ and all the values in this set can be realized. It is obvious that we have a formal similarity between the two cases. A relation between the Jones theory of subfactors and algebraic quantum field theory was found in [21]. Our classification results give further deeper relations between the two. (See [8] for a general theory of subfactors and related topics.)

In classification theory of subfactors, Ocneanu [26] has found a paragroup, which gives a combinatorial invariant for a subfactor through its representation theory. If the Jones index value is less than 4, the subfactor is of finite depth automatically, and this finite depth case is a special case of the amenable case which Popa's classification theorem [28] covers. We have shown in [17] that if the central charge value is less than 1, then the net of factors is automatically completely rational.

Wassermann's construction of the $SU(n)_k$ nets based on loop group representations gives the first examples of chiral algebraic conformal quantum field theories and they are completely rational. The finiteness of the μ-index for these nets was proved by Xu [34]. Xu also studied the coset and orbifold constructions in the setting of chiral algebraic conformal quantum field theory in [35, 36]. They give completely rational nets by [36, 23].

We briefly note that we have some formal similarity between our complete rationality and a condition in theory of vertex operator algebras, which is another mathematical approach to a chiral conformal field theory. They have a condition called *C_2-finiteness* introduced by Zhu [38], which is formally analogous to the above finiteness of the μ-index. See [10] for more details on vertex operator algebras. On a vertex operator algebra V, we have binary operations $a_{(n)}b$, $a, b \in V$, parameterized by integers n. The finiteness of the codimension $\dim V/V_{(-2)}V$ is called the C_2-finiteness condition. A vertex operator algebra V is said to be *rational* if every V-module is completely reducible, and this condition implies that V has only finitely many inequivalent simple modules. Zhu has proved that if we have the C_2-finiteness condition, then the modular group $SL(2, \mathbb{Z})$ acts on the space of characters of all the mutually inequivalent simple V-modules. This finiteness of the codimension and the above finiteness of the μ-index have some formal common similarity as follows.

1. The μ-index is also a certain multiplicative codimension.

2. Both the codimension $\dim V/V_{(-k)}V$ and the index μ_k can be defined for any positive integer k.

3. The above codimension and the index are trivial for $k = 1$.

4. If the above codimension and the index are finite for $k = 2$, then they are also finite for all positive integers k, and we obtain a unitary actions of the modular group $SL(2, \mathbb{Z})$ on certain natural finite dimensional spaces.

However, the action of $SL(2, \mathbb{Z})$ in the setting of vertex operator algebras is on the space of characters while its action in the setting of representation categories of a net of factors is on the intertwiner spaces, and we do not have any direct relation between the two situations. It would be very interesting to clarify this formal analogy.

4 α-induction, modular invariants, and classification

Here we explain our classification method for completely rational nets on S^1 with central charge less than 1.

Suppose we have a chiral algebraic conformal quantum field theory $\mathcal{A}(I)$, $I \subset S^1$, with central charge $c < 1$. Then the projective representation U of the diffeomorphism group gives a subnet as follows. For an interval $I \subset S^1$, we define $\mathcal{B}(I)$ be the von Neumann algebra generated by U_g, where g is a diffeomorphism which acts trivially outside of I. It is easy to see that this $\mathcal{B}(I)$ is a subalgebra of $\mathcal{A}(I)$ by the Haag duality

and $\mathcal{B}$ gives a subnet in the sense of [23]. (Note that the vacuum vector is not cyclic for $\mathcal{B}$.) We use a notation $\mathrm{Vir}_c(I)$ for this subnet and call it a Virasoro subnet with central charge c. This net is among the coset constructions due to Xu [35], which relies on A. Wassermann's construction of $SU(2)_k$-nets [32]. (See [5] for more on the Virasoro nets.) We have shown in [17] that the subfactor $\mathrm{Vir}_c(I) \subset \mathcal{A}(I)$ has a trivial relative commutant and finite index, which is a quite nontrivial fact.

For a net of subfactors $\mathrm{Vir}_c \subset \mathcal{A}$, we have a machinery of α-induction, which is analogous to a machinery of induction and restriction for representations of groups. For a DHR endomorphism λ of the smaller net Vir_c, we obtain an endomorphism $\alpha_\lambda^\pm$ of the larger net $\mathcal{A}$. This is "almost" a DHR endomorphism, but not completely. This operation is regarded as an extension of an endomorphism and depends on a choice $\pm$ of the braiding on the system of DHR endomorphisms of the smaller net. This method was defined by Longo-Rehren [24] and many interesting properties and examples were studied by Xu [33] and Böckenhauer-Evans [1]. Ocneanu [27] had a graphical method based on a quite different motivation, and it was unified with the theory of α-induction by us in [2, 3, 4]. One of the main results in [2] is that if we define a matrix Z by $Z_{\lambda\mu} = \dim \mathrm{Hom}(\alpha_\lambda^+, \alpha_\mu^-)$ for irreducible DHR endomorphisms λ, μ of the smaller net, then this matrix is in the commutant of the unitary representation of $SL(2,\mathbb{Z})$ arising from the braiding on the system of DHR endomorphisms of the smaller net. Obviously, each entry of Z is a nonnegative integer and we have $Z_{00} = 1$ for the vacuum representation denoted by 0. Such a matrix Z is called a *modular invariant* (of the unitary representation of $SL(2,\mathbb{Z})$).

It is easy to see that for a given unitary representation of $SL(2,\mathbb{Z})$, we have only finitely many modular invariants, and this finite number is often quite small in concrete examples. For the Virasoro net Vir_c, the corresponding modular invariants have been completely classified by Cappelli-Itzykson-Zuber [6] and they are labeled with pairs of A-D-E Dynkin diagrams with difference of the Coxeter numbers being 1. Also it is fairly easy to see in our current context that we have only so-called type I modular invariants in the classification of [6] where we have only A_n, D_{2n}, E_6, E_8 diagrams. In this way, starting with a chiral algebraic conformal quantum field theory $\mathcal{A}$ with central charge less than 1, we obtain a type I modular invariant matrix Z in the classification list of [6] labeled with pairs of the A_n-D_{2n}-$E_{6,8}$ Dynkin diagrams with difference of the Coxeter numbers being 1. Our main result in [17] with R. Longo is that this correspondence gives a complete classification of chiral algebraic conformal quantum field theories. Note that we have no reason, a priori, to believe or expect that this correspondence from a conformal field theory to a matrix in a certain list is injective or surjective, but we have proved both injectivity and surjectivity of this correspondence.

Our classification list is as follows.

1. Virasoro nets with central charge $c = 1 - 6/m(m+1)$.

2. Their simple current extensions of index 2.

3. The exceptional net labeled with (E_6, A_{12}).

4. The exceptional net labeled with (E_8, A_{30}).

5. The exceptional net labeled with (A_{10}, E_6).

6. The exceptional net labeled with (A_{28}, E_8).

The first two of the above exceptional ones are realized as the coset constructions for $SU(2)_{11} \subset SO(5)_1 \otimes SU(2)_1$ and $SU(2)_{29} \subset (G_2)_1 \otimes SU(2)_1$. They were first considered by Böckenhauer-Evans [1, II, Subsection 5.2] as possible candidates realizing the corresponding modular invariants in the Cappelli-Itzykson-Zuber list, but they were unable to show that these coset constructions indeed produce the desired modular invariants. With our complete classification, it is easy to identify these cosets with the above two in our list.

Recently, Köster [20] identified the third exceptional net in the above list , (A_{10}, E_6), with the two cosets $SU(9)_2 \subset (E_8)_2$ and $(E_8)_3 \subset (E_8)_2 \otimes (E_8)_1$, assuming that the local conformal nets $(E_8)_k$ have the expected WZW-fusion rules. The last one, (A_{28}, E_8), does not seem to be a coset nor an orbifold, and it appears to be a genuine new example.

Carpi [5] and Xu [37] recently obtained certain classification results of chiral algebraic conformal quantum field theories with central charge equal to 1, independently.

5 Classification of 2-dimensional theories and 2-cohomology

We now explain how to obtain a classification of full algebraic conformal quantum field theories with central charge less than 1, using the results in the previous section. This is our joint work with R. Longo [18]. As we mentioned before, our strategy is to study a subfactor $\mathcal{A}_+(I)\otimes\mathcal{A}_-(J) \subset \mathcal{A}(I\times J)$, where $\mathcal{A}(I\times J)$ is a given full algebraic conformal quantum field theory with central charge 1. By the classification list in the previous section, we have a complete information on the chiral ones $\mathcal{A}_\pm$.

We now assume the so-called parity symmetry condition for a full algebraic conformal quantum field theory $\mathcal{A}$, which in particular implies that $\mathcal{A}_+$ and $\mathcal{A}_-$ are isomorphic and they contain the same Vir_c. Then the dual canonical endomorphism for the subfactor $\mathrm{Vir}_c(I) \otimes \mathrm{Vir}_c(J) \subset \mathcal{A}(I \times J)$ gives a decomposition $\bigoplus_{\lambda,\mu} Z_{\lambda\mu}\lambda \otimes \mu$, where λ, μ are irreducible DHR endomorphisms of Vir_c. In a more general setting, the following was conjectured by Rehren and proved by Müger [25].

Theorem 5.1. *Under the above conditions, the following are equivalent.*

1. *The net $\mathcal{A}$ has only the trivial representation theory.*

2. *The μ-index of the net $\mathcal{A}$ is 1.*

3. *The matrix Z above is a modular invariant.*

In this way, we obtain a modular invariant Z for Vir_c in the classification list of Cappelli-Itzykson-Zuber [6] from a full algebraic conformal quantum field theory $\mathcal{A}$ with parity symmetry and trivial representation theory. Then we can prove as in [18]

that a full algebraic conformal quantum field theory $\mathcal{A}$ with parity symmetry has only the trivial representation theory if and only if it is maximal with respect to extensions. Thus we obtain a modular invariant in [6] from full algebraic conformal quantum field theory $\mathcal{A}$ with parity symmetry and maximality. Our main result in [18] with R. Longo shows that this gives a bijective correspondence. Note that the modular invariants in [6] are labeled with pairs of A-D-E Dynkin diagrams with difference of the Coxeter numbers being 1 as before, but we now do not have a restriction to so-called type I modular invariants, so the Dynkin diagrams D_{2n+1} and E_7 do appear.

Surjectivity of this correspondence is not difficult by Rehren's result [30], together with our previous analysis in [3, 17].

To prove injectivity of this correspondence, we need to study the subfactor $\mathcal{A}_+(I)\otimes\mathcal{A}_-(J)\subset\mathcal{A}(I\times J)$, where we have a natural identification of $\mathcal{A}_+(I)$ and $\mathcal{A}_-(J)$ and the dual canonical endomorphism decomposes in the form of $\bigoplus\lambda\otimes\lambda$, where λ is in a system of irreducible endomorphisms. In subfactor theory, such a subfactor was first studied by Ocneanu [26] under the name of the asymptotic inclusion and it plays a role of the quantum double construction. (See [8] and also [4] for more details.) Popa [29] gave a quite general construction and named it a symmetric enveloping algebra. Here we use a formulation of Longo-Rehren [24] where they gave a specific system of Q-system in the sense of Longo [22].

Based on [24], we studied this type of subfactors in [18] and found that the Q-system of a general subfactor of this type has a "twist" arising from a 2-cocycle of the tensor category of the endomorphisms. This notion of a 2-cocycle for a tensor category is a generalization of a 2-cocycle for a finite group, so it does not vanish in general, but we have proved in [18] it always vanishes for the tensor categories of the representation categories of chiral algebraic conformal quantum field theories with central charge less than 1. This 2-cohomology vanishing gives the desired injectivity of the above correspondence from a full algebraic conformal quantum field theory to the modular invariants in [6].

With extra combinatorial work along the same line, we can drop the triviality condition of the representation theory and classify full algebraic conformal quantum field theories with parity symmetry and central charge less than 1 completely as in [18].

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References

[1] J. Böckenhauer, D. E. Evans, *Commun. Math. Phys.* **197**, 361 (1998), II **200**, 57 (1999), III **205**, 183 (1999).

[2] J. Böckenhauer, D. E. Evans, Y. Kawahigashi, *Commun. Math. Phys.* **208**, 429 (1999).

[3] J. Böckenhauer, D. E. Evans, Y. Kawahigashi, *Commun. Math. Phys.* **210**, 733 (2000).

[4] J. Böckenhauer, D. E. Evans, Y. Kawahigashi, *Publ. RIMS, Kyoto Univ.* **37**, 1 (2001).

[5] S. Carpi, math.OA/0306425, to appear in *Commun. Math. Phys.*

[6] A. Cappelli, C. Itzykson, J.-B. Zuber, *Commun. Math. Phys.* **113**, 1 (1987).

[7] S. Doplicher, R. Haag, J. E. Roberts, I *Commun. Math. Phys.* **23** 199 (1971), II **35**, 49 (1974).

[8] D. E. Evans, Y. Kawahigashi, *Quantum symmetries on operator algebras*, (Oxford University Press, Oxford, 1998).

[9] K. Fredenhagen, K.-H. Rehren, B. Schroer, I. *Commun. Math. Phys.* **125**, 201 (1989), II. *Rev. Math. Phys.* **Special issue**, 113 (1992).

[10] E. Frenkel, D. Ben-Zvi, *Vertex algebras and algebraic curves*, Math. Surv. Monog. **88** (Amer. Math. Soc. 2001).

[11] D. Friedan, Z. Qiu, S. Shenker, *Commun. Math. Phys.* **107**, 535 (1986).

[12] P. Goddard, A. Kent, D. Olive, *Commun. Math. Phys.* **103**, 105 (1986).

[13] D. Guido, R. Longo, *Commun. Math. Phys.* **181**, 11 (1996).

[14] R. Haag, *Local Quantum Physics*, (Springer-Verlag, Berlin-Heidelberg-New York, 1996).

[15] V. F. R. Jones, *Invent. Math.* **72**, 1 (1983).

[16] Y. Kawahigashi, math.OA/0211141.

[17] Y. Kawahigashi, R. Longo, math-ph/0201015, to appear in *Ann. Math.*

[18] Y. Kawahigashi, R. Longo, math-ph/0304022, to appear in *Commun. Math. Phys.*

[19] Y. Kawahigashi, R. Longo, M. Müger, *Commun. Math. Phys.* **219**, 631 (2001).

[20] S. Köster, math-ph/0303054.

[21] R. Longo, I *Commun. Math. Phys.* **126**, 217 (1989), II *Commun. Math. Phys.* **130**, 285 (1990).

[22] R. Longo, *Commun. Math. Phys.* **159**, 133 (1994).

[23] R. Longo, *Commun. Math. Phys.* **237**, 7 (2003).

[24] R. Longo, K.-H. Rehren, *Rev. Math. Phys.* **7**, 567 (1995).

[25] M. Müger, *Extensions and modular invariants of rational conformal field theories*, in preparation.

[26] A. Ocneanu, Quantized group, string algebras and Galois theory for algebras, in *Operator algebras and applications, Vol. 2*, ed D. E. Evans and M. Takesaki, (Cambridge University Press, Cambridge, 1988) p. 119

[27] A. Ocneanu, Operator algebras, topology and subgroups of quantum symmetry – construction of subgroups of quantum groups – (written by S. Goto and N. Sato), in *Taniguchi Conference in Mathematics Nara '98* Adv. Stud. Pure Math. **31**, (Math. Soc. Japan, 2000) p. 235

[28] S. Popa, *Acta Math.* **172**, 352 (1994).

[29] S. Popa, *Math. Res. Lett.* **1**, 409 (1994).

[30] K.-H. Rehren, *Commun. Math. Phys.* **211**, 395 (2000).

[31] V. G. Turaev, *Quantum Invariants of Knots and* 3*-manifolds*, (Walter de Gruyter, 1994).

[32] A. Wassermann, *Invent. Math.* **133**, 467 (1998).

[33] F. Xu, *Commun. Math. Phys.* **192**, 347 (1998).

[34] F. Xu, *Commun. Contemp. Math.* **2**, 307 (2000).

[35] F. Xu, *Commun. Math. Phys.* **211**, 1 (2000).

[36] F. Xu, *Proc. Nat. Acad. Sci. U.S.A.* **97**, 14069 (2000).

[37] F. Xu, math.QA/0303266.

[38] Y. Zhu, *J. Amer. Math. Soc.* **9**, 237 (1996).